Numerical convergence analysis of Cauchy problem with noisy data solved by minimizing an energy-like functional

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Introduction

The Cauchy problem considered here consists of solving a partial differential equation on a domain for which over-specified boundary conditions are given on a part of its boundary, which means to recover the missing boundary conditions on the remaining part of the boundary. The Cauchy problem is known to be ill-posed and an important numerical instability may occur during the numerical resolution of this kind of problem. It provides researchers with an interesting challenge to carry out numerical procedure approximating the solution of Cauchy problem in the particular case of noisy data.

Physical problem

- **Domain** $\Omega$ such as $\partial \Omega = \Gamma_m \cup \Gamma_u$ et $\Gamma_m \cap \Gamma_u = \emptyset$.
- **Data** : Dirichlet and Neumann boundary conditions on $\Gamma_m$.
- **Identification** : Contact, pressure, interface crack, inclusion material properties, fluid pressure, flux, temperature, ... on $\Gamma_u$.

![Fig.: Geometry example.](image)

Cauchy problem and data completion

Find $u$ such as

$$\begin{align*}
\nabla \cdot (k(x) \nabla u) &= f \text{ in } \Omega \\
k(x) \nabla u \cdot n &= \phi \text{ on } \Gamma_m \\
u &= T \text{ on } \Gamma_m \\
k(x) \nabla u \cdot n &= \eta, u = \tau \text{ on } \Gamma_u
\end{align*}$$

Find $(\eta, \tau)$ on $\Gamma_u$ such as there exists $u$ solution of

$$\begin{align*}
\nabla \cdot (k(x) \nabla u) &= f \text{ in } \Omega \\
k(x) \nabla u \cdot n &= \phi, u = T \text{ on } \Gamma_m \\
k(x) \nabla u \cdot n &= \eta, u = \tau \text{ on } \Gamma_u
\end{align*}$$

An ill-posed problem

- The existence of the solution is caution to a verification of a compatibility condition on data $\phi$ et $T$ which can hardly be explicitly formulated.
- This compatibility condition imply that the stability assumption is not satisfied in the sense that the dependence of the solution $u$ on the data $(\phi, T)$ is not continuous.

Energy-like minimization problem

- We introduce two distinct fields $u_1$ and $u_2$ solutions of two well posed problems, each of them with one data on $\Gamma_m$ and one unknown on $\Gamma_u$:

$$\begin{align*}
\nabla \cdot (k(x) \nabla u_1) &= f \text{ in } \Omega \\
u_1 &= T \text{ on } \Gamma_m \\
k(x) \nabla u_1 \cdot n &= \eta \text{ on } \Gamma_u \\
u_2 &= \tau \text{ on } \Gamma_u
\end{align*}$$

- Minimization problem

$$\begin{align*}
(\eta^*, \tau^*) &= \arg \min_{\eta, \tau} E(\eta, \tau) \\
E(\eta, \tau) &= \int_\Omega k(x)(\nabla u_1(\eta) - \nabla u_2(\tau))^2 \, d\Omega
\end{align*}$$

Noisy data, a priori error estimates and stopping criteria

- During the optimization process : the error reaches a minimal before increasing and the functional attains a minimal threshold.

![Fig.: Error evolution during the optimization procedure for different noise rates.](image)

![Fig.: Evolution of $E(\eta, \tau)$ during the optimization procedure for different noise rates.](image)

Minimum characterization

- This minimization problem is re-formulate like an optimal control problem which allows to characterize the minimum introducing the adjoint states $v_1$, $v_2$ and the corresponding adjoint problems:

$$\begin{align*}
\nabla \cdot (k(x) \nabla v_1) &= 0 \text{ in } \Omega \\
v_1 &= 0 \text{ on } \Gamma_m \\
k(x) \nabla v_1 \cdot n &= k(x) \nabla u_1 \cdot n = \phi - k(x) \nabla u_1, n \text{ on } \Gamma_u \\
v_2 &= 0 \text{ on } \Gamma_u
\end{align*}$$

- The gradient of the functional is then given by:

$$\nabla E(\eta, \tau) = \left(-2v_1, 2(\eta - k(x) \nabla u_2, n - k(x) \nabla v_2, n)\right)$$

- The aim is to propose a stopping criteria depending on the noise rate $\alpha$ to stop the minimization process just before numerical imprecision. Then we write a priori error estimates taking into account the noisy data in order to theoretically determine the threshold attained by the functional when the error is minimal. We propose then the following stopping criteria :

$$\left| E_j - E_{j-1} \right| \leq \frac{E_j}{\hat{E}_j} \left( \frac{\alpha^2}{1-\alpha} \right) \left( \| T^j \|^2 + \| \phi \|^2 \right)$$

where $E_j$ denote the value of the functional at the $j$-th iteration of the optimization algorithm and $(\phi^j, T^j)$ the noisy data.

Numerical results : stratified inner fluid example

We consider the reconstruction of temperature and flux in a pipeline of infinite length. We assume that the temperature does not depend on the longitudinal coordinate. The Cauchy data are generated by solving the following problem:

$$\begin{align*}
\nabla \cdot (k(T) u) &= 0 \text{ in } \Omega \\
k(T) \nabla u \cdot n + \alpha u &= T \text{ on } \Gamma
\end{align*}$$

where $T$ is constant on the outer boundary ($\Gamma_m$) and stratified on the inner boundary ($\Gamma_u$) of the pipe.

![Fig.: Exact and identified Dirichlet boundary conditions with a noise rate of 10%.](image)

![Fig.: Exact and identified Neumann boundary conditions with a noise rate of 10%.](image)

Exact and Identified fields

- **Conclusion**

  - The Energy-like method allows to consider the Cauchy problem as minimization problem depending on two well-posed problems.
  - Numerical analysis of the method permits to indentify an adequate stopping criteria depending on noise rate.
  - The stratified inner fluid example illustrates robustness and efficiency of the proposed stopping criteria in the case of singular data.

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